Cartan Subgroups and Generosity in $SL_2(\mathbb{Q}_p)$

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Abstract

We describe all Cartan subgroups of $SL_2(\mathbb{Q}_p)$. We show that the Cartan subgroup consisting of all diagonal matrices is generous and it is the only one up to conjugacy.

Keywords p-adic field ; Cartan subgroup ; generosity
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A subset X of a group G is *left-generic* if G can be covered by finitely many left-translates of X. We define similary right-genercity. If X is G-invariant, then left-genericity is equivalent to right-genericity. This important notion in model theory was particulary developped by B. Poizat for groups in stable theories [3]. For a group of finite Morley-rank and X a definable subset, generosity is the same as being of maximal dimension [3, lemme 2.5]. The term generous was introduced in [2] to show some conjuguation theorem. A definable subset X of a group G is *generous* in G if the union of its G-conjugates, $X^G = \{x^g \mid (x,g) \in X \times G\}$, is generic in G.

In an arbitrary group G, we define a *Cartan subgroup* H as a maximal nilpotent subgroup such that every finite index normal subgroup $X \trianglelefteq H$ is of finite index in its normalizer $N_G(X)$. In connected reductive algebraic groups over an algebraically closed fields, the maximal torus is typically an example of a Cartan subgroup. Moreover it is the only one up to conjugation and it is generous. It has been remarked in [1] that, in the group $SL_2(\mathbb{R})$, the Cartan subgroup consisting of diagonal matrices is also generous. But it has also been remarked that in the case of $SL_2(\mathbb{R})$, there exists another Cartan subgroup, namely $SO_2(\mathbb{R})$, which is not generous.

We will discuss here some apparently new remarks of the same kind in $SL_2(\mathbb{Q}_p)$. First we describe all Cartan subgroups of $SL_2(\mathbb{Q}_p)$. After we show that the Cartan

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subgroup consisting of diagonal matrices is generous and it is the only one up to conjugacy.

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Description of Cartan subgroups up to conjugacy

We note $v_p : \mathbb{Q}_p \longrightarrow \mathbb{Z} \bigcup \{+\infty\}$ the p-adic-valuation, and $ac : \mathbb{Q}_p^\times \longrightarrow \mathbb{F}_p$ the angular component defined by $ac(x) = res(p^{-v_p(x)}x)$ where $res : \mathbb{Q}_p \longrightarrow \mathbb{F}_p$ is the residue map.

With this notations, if $p \neq 2$, an element $x \in \mathbb{Q}_p^\times$ is a square if and only if $v_p(x)$ is even and $ac(x)$ is a square in $\mathbb{F}_p$. For $p = 2$, an element $x \in \mathbb{Q}_2$ can be written $x = 2^n u$ with $n \in \mathbb{Z}$ and $u \in \mathbb{Z}_2^\times$, then x is a square if n is even and $u \equiv 1 (\mathrm{mod}\ 8)$ [4].

Fact 1 ([4])**.** *If $p \neq 2$, the group $\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, it has for representatives $\{1, u, p, up\}$, where $u \in \mathbb{Z}_p^\times$ is such that $ac(u)$ is not a square in $\mathbb{F}_p$*

The group $\mathbb{Q}_2^\times/(\mathbb{Q}_2^\times)^2$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$, it has for representatives $\{\pm1, \pm2, \pm5, \pm10\}$.

For any prime p, and any δ in $\mathbb{Q}_p^\times \backslash (\mathbb{Q}_p^\times)^2$, we put :

$$Q_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in SL_2(\mathbb{Q}_p) \mid a \in \mathbb{Q}_p^\times \right\}$$
$$Q_\delta = \left\{ \begin{pmatrix} a & b \\ b\delta & a \end{pmatrix} \in SL_2(\mathbb{Q}_p) \mid a, b \in \mathbb{Q}_p \text{ and } a^2 - b^2\delta = 1 \right\}$$

Lemma 1.

$$\forall x \in Q_1 \backslash \{I, -I\} \quad C_{SL_2(\mathbb{Q}_p)}(x) = Q_1$$
$$\forall x \in Q_\delta \backslash \{I, -I\} \quad C_{SL_2(\mathbb{Q}_p)}(x) = Q_\delta$$

The checking of these equalities is left to the reader.

Proposition 1. *The groups Q_1 and Q_δ are Cartan subgroups of $SL_2(\mathbb{Q}_p)$*

Proof. One checks easily that Q_1 is abelian and the normalizer of Q_1 is :

$$N_{SL_2(\mathbb{Q}_p)}(Q_1) = Q_1 \cdot < \omega > \qquad \text{where} \qquad \omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

For X a subgroup of Q_1, if $g \in N_{SL_2(\mathbb{Q}_p)}(X)$ and $x \in X$, then, using lemma 1

$$Q_1 = C_{SL_2(\mathbb{Q}_p)}(x) = C_{SL_2(\mathbb{Q}_p)}(x^g) = C_{SL_2(\mathbb{Q}_p)}(x)^g = Q_1^g$$

It follows that $N_{SL_2(\mathbb{Q}_p)}(X) = N_{SL_2(\mathbb{Q}_p)}(Q_1) = Q_1 \cdot < \omega >$ and if X of finite index k in Q_1, then X is of index $2k$ in $N_{SL_2(\mathbb{Q}_p)}(X)$. We can see that for t in Q_1, $t^\omega = \omega^{-1} t \omega = t^{-1}$ thus $N_{SL_2(\mathbb{Q}_p)}(Q_1)' = Q_1^2$ and $[Q_1^2, \omega] = Q_1^2$, in particular $N_{SL_2(\mathbb{Q}_p)}(Q_1)$ is not nilpotent. By

the normalizer condition for nilpotent groups, if Q_1 is properly contained in a nilpotent group K, then $Q_1 < N_K(Q_1) \leq K$, here $N_K(Q_1) = Q_1 \cdot < \omega >$ which is not nilpotent, a contradiction. It finishes the proof that Q_1 is a Cartan subgroup.

For $\delta \in \mathbb{Q}_p^\times \backslash (\mathbb{Q}_p^\times)^2$, we check similary that the group Q_δ is abelian. Since for all subgroups X of Q_δ, $C_{SL_2(\mathbb{Q}_p)}(X) = Q_\delta$, it follows that $N_{SL_2(\mathbb{Q}_p)}(X) = N_{SL_2(\mathbb{Q}_p)}(Q_\delta) = Q_\delta$, and if X is of finite index in Q_δ then X is of finite index in its normalizer. By the normalizer condition for nilpotent groups, Q_δ is nilpotent maximal. $\square$

Proposition 2. $Q_1^{SL_2(\mathbb{Q}_p)} = \left\{ A \in SL_2(\mathbb{Q}_p) \mid tr(A)^2 - 4 \in (\mathbb{Q}_p^\times)^2 \right\} \bigcup \{I, -I\}$
$Q_\delta^{SL_2(\mathbb{Q}_p)} = \left\{ A \in SL_2(\mathbb{Q}_p) \mid tr(A)^2 - 4 \in \delta \cdot (\mathbb{Q}_p^\times)^2 \right\} \bigcup \{I, -I\}$

We put :

$$U = \left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \mid u \in \mathbb{Q}_p \right\} \bigcup \left\{ \begin{pmatrix} -1 & u \\ 0 & -1 \end{pmatrix} \mid u \in \mathbb{Q}_p \right\}$$

If $A \in SL_2(\mathbb{Q}_p)$ satisfies $tr(A)^2 - 4 = 0$, then either $tr(A) = 2$ or $tr(A) = -2$, and A is a conjugate of a element of U. In this case, A is said *unipotent*. It follows, from Proposition 2 :

Corollary 3. *We have the following partition :*

$$SL_2(\mathbb{Q}_p) \backslash \{I, -I\} = (U \backslash \{I, -I\})^{SL_2(\mathbb{Q}_p)} \sqcup (Q_1 \backslash \{I, -I\})^{SL_2(\mathbb{Q}_p)} \sqcup \bigsqcup_{\delta \in \mathbb{Q}_p^\times / (\mathbb{Q}_p^\times)^2} (Q_\delta \backslash \{I, -I\})^{SL_2(\mathbb{Q}_p)}$$

Remark. *If δ and δ' in $\mathbb{Q}_p^\times$ are in the same coset of $(\mathbb{Q}_p^\times)^2$, then, by Proposition 2, if $x' \in Q_{\delta'}$, then there exists $x \in Q_\delta$ and $g \in SL_2(\mathbb{Q}_p)$, such that $x' = x^g$, thus, by lemma 1, $Q_{\delta'} = C_{SL_2(\mathbb{Q}_p)}(x') = C_{SL_2(\mathbb{Q}_p)}(x)^g = Q_\delta^g$. Therefore the Corollary 3 makes sense.*

Proof of Proposition 2. • If $A \in Q_1^{SL_2(\mathbb{Q}_p)}$, then there exists $P \in SL_2(\mathbb{Q}_p)$ such that

$$A = P \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} P^{-1}$$

with $a \in \mathbb{Q}_p^\times$. We have $tr(A) = a + a^{-1}$, so $tr(A)^2 - 4 = (a + a^{-1})^2 - 4 = (a - a^{-1})^2$ and $tr(A)^2 - 4 \in (\mathbb{Q}_p^\times)^2$.

Conversely , let A be in $SL_2(\mathbb{Q}_p)$ with $tr(A)^2 - 4$ a square. The caracteristic polynomial is $\chi_A(X) = X^2 - tr(A)X + 1$ and its discriminant is $\Delta = tr(A)^2 - 4 \in (\mathbb{Q}_p^\times)^2$, so χ_A has two distinct roots in $\mathbb{Q}_p$ and A is diagonalizable in $GL_2(\mathbb{Q}_p)$. There is $P \in GL_2(\mathbb{Q}_p)$, and $D \in SL_2(\mathbb{Q}_p)$ diagonal such that $A = PDP^{-1}$. If

$$P = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

we put

$$\tilde{P} = \begin{pmatrix} \frac{\alpha}{det(P)} & \beta \\ \frac{\gamma}{det(P)} & \delta \end{pmatrix}$$

and we have $\tilde{P} \in SL_2(\mathbb{Q}_p)$ and $A = \tilde{P}D\tilde{P}^{-1} \in Q_1^{SL_2(\mathbb{Q}_p)}$.

• If A is in $Q_\delta^{SL_2(\mathbb{Q}_p)} \backslash \{I, -I\}$, then $tr(A) = 2a$ and there exists $b \neq 0$ such that $a^2 - b^2\delta = 1$. So $tr(A)^2 - 4 = 4a^2 - 4 = 4(b^2\delta + 1) - 4 = (2b)^2\delta \in \delta \cdot (\mathbb{Q}_p^\times)^2$

Conversely we proceed as in the real case and the root $i \in \mathbb{C}$. The discriminant of χ_A, $\Delta = tr(A)^2 - 4$ is a square in $\mathbb{Q}_p(\sqrt{\delta})$, and the caracteristic polynomial χ_A has two roots in $\mathbb{Q}_p(\sqrt{\delta})$: $\lambda_1 = \alpha + \beta\sqrt{\delta}$ and $\lambda_2 = \alpha - \beta\sqrt{\delta}$ (with $\alpha, \beta \in \mathbb{Q}_p$). For the two eigen values λ_1 and λ_2, A has eigen vectors :

$$v_1 = \begin{pmatrix} x + y\sqrt{\delta} \\ x' + y'\sqrt{\delta} \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} x - y\sqrt{\delta} \\ x' - y'\sqrt{\delta} \end{pmatrix}$$

In the base $\{(x, x'), (y, y')\}$, the matrix A can be written :

$$\begin{pmatrix} a & b \\ b\delta & a \end{pmatrix}$$

with $a, b \in \mathbb{Q}_p$. As above, we can conclude that there exists $P \in SL_2(\mathbb{Q}_p)$ such that :

$$A = P \begin{pmatrix} a & b \\ b\delta & a \end{pmatrix} P^{-1}$$

□

Theorem 4. *The subgroups Q_1 and Q_δ (for $\delta \in \mathbb{Q}_p^\times \backslash (\mathbb{Q}_p^\times)^2$) are the only Cartan subgroups up to conjugacy of $SL_2(\mathbb{Q}_p)$*

Remark. *By Fact 1and the previous remark, for $p \neq 2$ there are four Cartan subgroup up to conjugacy in $SL_2(\mathbb{Q}_p)$, and for $p = 2$, there are eight.*

Proof. For the demonstration we note $G = SL_2(\mathbb{Q}_p)$ and B the following subgroup of $SL_2(\mathbb{Q}_p)$:

$$B = \left\{ \begin{pmatrix} t & u \\ 0 & t^{-1} \end{pmatrix} \mid t \in \mathbb{Q}_p^\times, u \in \mathbb{Q}_p \right\}$$

With these notations, we can easily check for $g \in U \backslash \{I, -I\}$ that $C_G(g) = U$ and $N_G(U) = B$. Moreover it is kown that every $q \in B$ can be written as $q = tu$ where $t \in Q_1$ and $u \in U$.

Consider K a Cartan subgroup of $SL_2(\mathbb{Q}_p)$. We will show that K is a conjugate of Q_1 or Q_δ (for $\delta \in \mathbb{Q}_p^\times \backslash (\mathbb{Q}_p^\times)^2$). First we prove K cannot contain a unipotent element other than I or $-I$. Since a conjugate of a Cartan subgroup is still a Cartan subgroup, it suffice to show that $K \cap U = \{I, -I\}$.

In order to find a contradiction, let $u \in K$ be a element of U different of I or $-I$, u is in $K \cap B$. If $\alpha \in N_G(K \cap B)$, then we have that $u^\alpha \in K \cap B$, and since $tr(u^\alpha) = tr(u) = \pm 2$, u^α is still in U. Therefore $U = C_G(u) = C_G(u^\alpha) = C_G(u)^\alpha = U^\alpha$ and so α is in $N_G(U) = B$. It follows $N_K(K \cap B) \leq B$ and finally $N_K(K \cap B) = K \cap B$. By the normalizer condition $K \cap B$ cannot be proper in K, then $K \leq B$.

It is known (see for example [5, Lemma 0.1.10]) that if K is a nilpotent group and $H \trianglelefteq K$ a non trivial normal subgroup, then $H \cap Z(K)$ is not trivial. If we assume that $K \nleq U$, since $K \leq B = N_G(U)$, $K \cap U$ is normal in K, and so $K \cap U$ contains an element x of the center $Z(K)$. For $q \in K \backslash U$, there are $t \in Q_1 \backslash \{I, -I\}$ and $u \in U$ such that $q = tu$. We have $[x, q] = I$ so $[x, t] = I$, that is impossible because $C_G(x) = U$. Therefore $K \leq U$. Since K is maximal nilpotent and U abelian, $K = U$. But U is not a cartan subgroup, because it is of infinite index in its normalizer B. A contradiction.

Since K does not contain a unipotent element, K intersects a conjugate of Q_1 or Q_δ (for $\delta \in \mathbb{Q}_p^\times \backslash (\mathbb{Q}_p^\times)^2$) by Corollary 3, we note Q this subgroup. Let us show that $K = Q$. Let be x in $K \cap Q$, and $\alpha \in N_K(K \cap Q)$, then $x^\alpha \in Q$, and, by lemma 1, $Q = C_G(x^\alpha) = C_G(x)^\alpha = Q^\alpha$. Thus $\alpha \in N_G(Q)$, and $N_K(K \cap Q) \leq N_G(Q)$.

1rst case Q is a conjugate of Q_1, then $N_G(Q) = Q \cdot < w' >$ where $w' = w^g$ if $Q = Q_1^g$. We have also $w'^2 \in Q$ and $t^{w'} = t^{-1}$ for $t \in Q$. If $w' \in K$ then $P = Q \cdot < w' > \cap K$ is a subgroup of K, but $P' = P^2$ which is K-invariant and P is not nilpotent. A contradiction, so $w' \notin K$. Then $N_K(Q \cap K) \leq Q \cap K$, it follows by normalizer condition that $K \leq Q$, and by maximality of K, $K = Q$.

2nd case Q is a conjuguate of Q_δ (for $\delta \in \mathbb{Q}_p^\times \backslash (\mathbb{Q}_p^\times)^2$), then $N_G(Q) = Q$. It follows similarly that $K = Q$.

□

Generosity of the Cartan subgroups

Our purpose is now to show the generosity of the Cartan subgroup Q_1. It follows from the next more general proposition :

Proposition 5. *1. The set $W = \{A \in SL_2(\mathbb{Q}_p) \mid v_p(tr(A)) < 0\}$ is generic in $SL_2(\mathbb{Q}_p)$.*

2. The set $W' = \{A \in SL_2(\mathbb{Q}_p) \mid v_p(tr(A)) \geq 0\}$ is not generic in $SL_2(\mathbb{Q}_p)$.

Proof. 1. We consider the matrices :

$$A_1 = I, \quad A_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \quad \text{and} \quad A_4 = \begin{pmatrix} 0 & -b^{-1} \\ b & 0 \end{pmatrix}$$

with $v_p(a) > 0$ and $v_p(b) > 0$.

We show that $SL_2(\mathbb{Q}_p) = \bigcup_{i=1}^4 A_i W$. Suppose there exists

$$M = \begin{pmatrix} x & y \\ u & v \end{pmatrix} \in SL_2(\mathbb{Q}_p)$$

such that $M \notin \bigcup_{i=1}^4 A_i W$.

Since $M \notin A_1W \bigcup A_2W$, we have $x+v=\varepsilon$ and $y-u=\delta$ with $v_p(\varepsilon)\geq 0$ and $v_p(\delta)\geq 0$. Since $M \notin A_3W$, we have $ax+a^{-1}v=\eta$ with $v_p(\eta)\geq 0$. We deduce $a(\varepsilon-v)+a^{-1}v=\eta$ and $v=\frac{\eta-a\varepsilon}{a^{-1}-a}$. Similarly, it follows from $M\notin A_4W$ that $u=\frac{\theta-b\delta}{b^{-1}-b}$ with some θ such that $v_p(\theta)\geq 0$.

Since $v_p(a)>0$, we have $v_p(a+a^{-1})<0$. From $v_p(\eta-a\varepsilon)\geq min\{v_p(\eta);v_p(a\varepsilon)\}\geq 0$, we deduce that $v_p(v)=v_p(\frac{\eta-a\varepsilon}{a+a^{-1}})=v_p(\eta-a\varepsilon)-v_p(a+a^{-1})>0$. Similarly $v_p(u)>0$. It follows that $v_p(x)=v_p(\varepsilon-v)\geq 0$ and $v_p(y)\geq 0$.

Therefore $v_p(det(M))=v_p(xv-uy)\geq min\{v_p(xv),v_p(uy)\}>0$ and thus $det(M)\neq 1$, a contradiction .

2. We show that the family of matrices $(M_x)_{x\in\mathbb{Q}_p^\times}$ cannot be covered by finitely many $SL_2(\mathbb{Q}_p)$-translates of W', where :

$$M_x=\begin{pmatrix} x & 0\\ 0 & x^{-1}\end{pmatrix}$$

Let $A=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{Q}_p)$. Then $tr(A^{-1}M_x)=dx+ax^{-1}$. If $v_p(x)>max\{|v_p(a)|,|v_p(d)|\}$ then $v_p(tr(A^{-1}M_x))<0$ and $M_x\notin AW'$.

Therefore for every finite family $\{A_j\}_{i\leq n}$, there exist $x\in\mathbb{Q}_p$ such that $M_x\notin\bigcup_{j=1}^n A_jW'$. $\square$

Remark. *We remark that the sets W and W' form a partition of $SL_2(\mathbb{Q}_p)$. There are both definable because the valuation v_p is definable in $\mathbb{Q}_p$.*

Lemma 2. *$W\subseteq Q_1^{SL_2(\mathbb{Q}_p)}$ and for $\delta\in\mathbb{Q}_p^\times\backslash(\mathbb{Q}_p^\times)^2$, $Q_\delta^{SL_2(\mathbb{Q}_p)}\subseteq W'$*

Proof. Let be $A\in SL_2(\mathbb{Q}_p)$ with $v_p(tr(A))<0$.
For $p\neq 2$, since $v_p(tr(A))<0$, $v_p(tr(A)^2-4)=2v_p(tr(A))$ and $ac(tr(A)^2-4)=ac(tr(A)^2)$, so $tr(A)^2-4$ is a square in $\mathbb{Q}_p$.
For $p=2$, we can write $tr(A)=2^nu$ with $n\in\mathbb{Z}$ and $u\in\mathbb{Z}_p^\times$. Then $tr(A)^2-4=2^{2n}(u^2-4\cdot 2^{-2n})$. Since $n\leq -1$, $u^2-4\cdot 2^{-2n}\equiv u^2\equiv 1(\text{mod } 8)$, so $tr(A)^2-A\in(\mathbb{Q}_2^\times)^2$.

In all cases, by the proposition 2, $W\subseteq Q_1^{SL_2(\mathbb{Q}_p)}$ and ,by complementarity, $Q_\delta^{SL_2(\mathbb{Q}_p)}\subseteq W'$. $\square$

We can now conclude with the following corollary, similar to [1, Remark 9.8] :

Corollary 6. *1. The Cartan subgroup Q_1 is generous in $SL_2(\mathbb{Q}_p)$.*

2. The Cartan subgroups Q_δ (for $\delta\in\mathbb{Q}_p^\times\backslash(\mathbb{Q}_p^\times)^2$) are not generous in $SL_2(\mathbb{Q}_p)$.

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